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A comparison of optimal control methods for minimum fuel cruise at constant altitude and course with fixed arrival time

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Abstract

The current lack of efficiency in the use of airspace, enforced by the economic crisis an the increasing competitiveness among air transport companies are drivers for a renewed interest for the application of optimization techniques to find answers to overcame current inefficiencies.

The main objective of the present paper is to assess and compare different optimal aircraft trajectories techniques applied to the minimum fuel cruise problem at constant altitude and course with fixed arrival time, International Standard Atmosphere and without wind. Four trajectory optimization methods have been used: Hermite-Simpson , 5th degree Gauss-Lobatto and Radau pseudospectral collocation methods and the singular arc solution.

Hermite-Simpson and 5th degree methods have been programmed in Ampl modeling language with an IPOPT solver and Radau pseudospectral method using gpops matlab tool with SNOPT solver.

5th degree Gauss-Lobatto collocation method gives the less fuel consumption solution followed by Radau pseudospectral, Hermite-Simpson and singular arc. In considering the program execution time, Hermite-Simpson collocation method is the fastest method followed by 5th degree and Radau pseudospectral. Also, taking into account the time for developing the program code the Radau pseudospectral is the most user friendly. Moreover, it has been observed that increasing the sample points in the Hermite-Simpson and 5th degree, the solution converge to the minimum fuel consumption solution. On the other hand, gpops does not show much sensitivity to the number of sample points.

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1. Introduction

In the future Air Traffic Management (ATM) system, the trajectory becomes the fundamental element of a new set of operating procedures collectively referred to as Trajectory-Based Operations (TBO) [1]. The underlying idea behind TBO is the concept of business trajectory. The business trajectory is the trajectory that will meet best airline business interests. This business interests may be, for instance, minimum duration, minimum consumption, or minimum operational cost. The TBO concept of operations and the notion of business trajectory will result in more

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efficient 4D trajectories, that will be necessarily flown under the presence of constraints due to, for instance, airport operations or Air Traffic Control (ATC) intervention. Any modification in that trajectory will result in a change in the cost effectiveness of the operation. Thus, the future ATM system should modify the business trajectory as little as possible. Furthermore, the necessary tactical intervention will be limited to exceptions, so the development of techniques for strategic planning of business 4D trajectories will be key, resulting in significant fuel savings for airlines. Effective flight planning cannot only reduce fuel costs, but also time-based costs and lost revenue from payload that can not be carried, simply by choosing efficient routes and altitudes, speed, and the optimal amount of departure fuel.

The flight planning problem can be regarded as a trajectory optimization problem. The trajectory optimization problem can be studied as an optimal control problem of a dynamic system in which the goal is to find the trajectory and the corresponding control inputs that steer the state of the system between two configurations satisfying a set of constraints on the state and/or control variables while minimizing an objective functional.

Typically, optimal control problems are highly non-linear and it is very difficult to find an analytical solution even for the simplest cases. The common practice is to use numerical methods to obtain solutions. There are three fundamental approaches to numerically solving continuos time optimal control problems: Dynamic Programming (DP) methods, whose optimality criteria in continuous time is based on the Hamilton-Jacobi-Bellman partial differential equation [2]; indirect methods, that rely on the necessary conditions of optimality that can be derived from the Pontryagin's maximum principle [3]; direct methods, that are based on a finite dimensional parameterization of the infinite-dimensional problem [4].

In the scope of commercial aircraft trajectory optimization using optimal control, different methods have been used. For instance, in [5] the authors analyze the optimal performance of an aircraft in cruise conditions solving the problem as a singular arc. Also, the singular arc trajectory was analyzed for climb performances [6]. Dynamic Programming has been also used very recently to solve the vertical profile [7]. More complex problems have been solved using different direct methods.

For instance, Hermite-Legendre-Gauss-Lobatto collocation methods have been used to solve commercial aircraft trajectory planning problems [8, 9, 10, 11]. Also, recent advances have been made in pseudospectral collocation methods [12, 13]. In particular, the software package GPOPS [14] implements Gauss and Radau pseudospectral methods. Its interface with matlab makes it a very powerful and user friendly tool.

However, there is still a lack of knowledge in terms of analyzing the performances of the different methods within the particularities of commercial aircraft trajectory optimization.

Therefore, the main contribution of this paper is to present a comparison of different optimal control methods applied the problem of minimum fuel cruise at constant altitude and heading with fixed arrival time. The different methods are discussed and solutions to the problem are presented using them. Namely, the problem is solved as a singular arc problem, using Hermite-Legendre-Gauss-Lobatto direct collocation methods (Hermite-Simpson and 5th degree), and using the software GPOPS.

Te paper is structure as follows. First, in Section 2, we state the optimal control problem and present the optimality conditions. In Section 3, the most common numerical methods to solve such problem are described. The problem of minimum fuel cruise at constant altitude and heading with fixed arrival time is then presented in Section 4. Subsequently, results are reported and discussed in Section 5. Finally, some conclusions and future directions of research are drawn in Section 6.

2. Optimal Control Problem

Control theory is a discipline that studies the behavior of dynamical systems with control inputs. In general, the aim is to control the state of the dynamical system in some prescribed manner. The goal of optimal control theory is to determine the control input that will cause a system to achieve the control objectives, satisfying the constraints, and at the same time optimize some performance criterion.

The trajectory planning problem is in general solved following an open loop terminal control problem. This strategy allows all the constraints acting on the dynamical system, including the dynamic constraints, to be taken into account in such a way that the resulting trajectory is admissible. However this problem has an infinite number of solutions. To eliminate this redundancy optimal control techniques can be used to select only one of them, the trajectory that optimize a given criterion. Once an admissible trajectory or the optimal one has been found, a closed loop tracking control strategy is in general used to follow it.

The optimal control problem can be stated as follows:

Problem 1 (***Optimal Control Problem***).

$$\begin{aligned}
&\min J(t, x(t), u(t), l) = E(t^F, x(t^F)) + \int_{t^I}^{t^F} L(x(t), u(t), l)dt;\\
&\text{subject to:}\\
&\dot{x}(t) = f(x(t), u(t), l),\ \text{dynamic equations;}\\
&0 = g(x(t), u(t), l),\ \text{algebraic equations;}\\
&x(t^I) = x^I,\ \text{initial boundary conditions;}\\
&\psi(x(t^F)) = 0,\ \text{terminal boundary conditions;}\\
&\phi_l \le \phi[x(t), u(t), p] \le \phi_u,\ \text{path constraints.}
\end{aligned} \tag{OCP}$$

Variable $t \in [t^I, t^F] \subset \mathbb{R}$ represents time and $l \in \mathbb{R}^{n_l}$ is a vector of parameters. Notice that the initial time t^I is fixed and the final time t^F might be fixed or left undetermined. $x(t) : [t^I, t^F] \mapsto \mathbb{R}^{n_x}$ represents the state variables. $u(t) : [t^I, t^F] \mapsto \mathbb{R}^{n_u}$ represents the control functions, also referred to as control inputs, assumed to be measurable. The objective function $J : [t^I, t^F] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$ is given in Bolza form. It is expressed as the sum of the Mayer term $E(t^F, x(t^F))$ and the Lagrange term $\int_{t^I}^{t^F} L(x(t), u(t), l)dt$. Functions $E : [t^I, t^F] \times \mathbb{R}^{n_x} \to \mathbb{R}$ and $L : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}$ are assumed to be twice differentiable. The system is a DAE system in which the right hand side function of the differential set of equations $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{n_x}$ is assumed to be piecewise Lipschitz continuous, and the derivative of the algebraic right hand side function $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{n_z}$ with respect to z is assumed to be regular. $x^I \in \mathbb{R}^{n_x}$ represents the vector of initial conditions given at the initial time t^I and the function $\psi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_q}$ provides the terminal conditions at the final time and it is assumed to be twice differentiable. The system must satisfy algebraic path constraints given by the function $\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_l} \to \mathbb{R}^{n_\phi}$ with lower bound $\phi_l \in \mathbb{R}^{n_\phi}$ and upper bound $\phi_u \in \mathbb{R}^{n_\phi}$. Function ϕ is assumed to be twice differentiable.

2.1. Pontryagin's necessary conditions

2.1.1. Unconstrained problems

For the sake of clarity, let us consider a simpler problem with no path constraints, no algebraic constraints, nor dependence on the vector of parameters p, i.e., let us consider the following unconstrained optimal control problem:

$$\min J(t, x(t), u(t)) = \Phi(t_F, x(t_F)) + \int_{t_I}^{t_F} L(t, x(t), u(t))dt. \tag{1a}$$

Subject to:

$$\dot{x}(t) = f(t, x(t), u(t)),\ \text{dynamic constraints;} \tag{1b}$$

$$\psi(x(t_F)) = 0,\ \text{final boundary condition;} \tag{1c}$$

where the initial conditions $x(t_I) = x_I$ are given at the fixed initial time t_I, and the final time t_F is undetermined. The designators $J, \Phi, L, f, \psi, t, x, u$ are as in problem (OCP) besides changes due to not including the vector of parameters p.

The problem (1) is to find the admissible control functions $u^*(t)$ that minimize (or maximize) the performance index $J(t, x(t), u(t))$ in equation (1a) and fulfill the set of differential equations (1b), final boundary conditions (1c) and the initial conditions $x(t_I) = x_I$.

Let us adjoin to $J(t, x(t), u(t))$ the system differential equation (1b) with functions $\lambda(t) : [t_I, t_F] \to \mathbb{R}^{n_x}$, and the final boundary condition (1c) with multipliers $\mu \in \mathbb{R}^{n_q}$. $\lambda(t)$ are assumed to be continuously differentiable functions. Then, we can define the Lagrangian of the problem as:

Definition 2.1 (***Lagrangian***). *The Lagrangian of the unconstrained optimal control problem (1) is a continuously differentiable function $\mathcal{L} : [t_I, t_F] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_q} \times \mathbb{R}^{n_x} \to \mathbb{R}$:*

$$\mathcal{L}(t, x(t), u(t), \mu, \lambda(t)) = \Big(\Phi(t_F, x(t_F)) + \mu^T \psi(x(t_F))\Big) + \int_{t_I}^{t_F} \Big[L(t, x(t), u(t)) + \lambda^T(t)(\dot{x} - f(t, x(t), u(t)))\Big] dt, \tag{2}$$

where the variables $\lambda_i(t), i = 1, \ldots, n_x$ are the adjoint variables or costates for the dynamic constraints $\dot{x} - f(t, x(t), u(t)) = 0$, and $\mu \in \mathbb{R}^q$ are Lagrange multipliers associated to the number of final constraints $\psi(x(t_F)) \in \mathbb{R}^q$.

Let us define the Hamiltonian $\mathcal{H}$:

Definition 2.2 (***Hamiltonian***)**.** *The* Hamiltonian *of the unconstrained optimal control problem* (1) *is a scalar function* $\mathcal{H} : [t_I, t_F] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \to \mathbb{R}$ *defined by*

$$\mathcal{H}(t, x(t), u(t), \lambda(t)) = L(t, x(t), u(t)) + \lambda^T(t) f(t, x(t), u(t)). \tag{3}$$

Let us also define the auxiliary function $\varphi[t_I, t_F] \times \mathbb{R}^{n_x} \to \mathbb{R}$:

$$\varphi(t, x) = \Phi(t, x(t)) + \mu^T \psi(x(t)). \tag{4}$$

Integrating by parts the last term on the right side in equation (2), it yields:

$$\mathcal{L}(t, x(t), u(t), \mu, \lambda(t)) = [\varphi(t, x(t))]_{t=t_F} - \lambda^T(t_F)x(t_F) + \lambda^T(t_I)x(t_I) + \int_{t_I}^{t_F} \left[\mathcal{H}(t, x(t), u(t), \lambda(t)) + \dot{\lambda}^T(t)x(t)\right] dt. \tag{5}$$

Necessary conditions for optimality of solution trajectories trajectories $(x^*(t), u^*(t))$, $t \in I = [t_I, t_F]$ can be derived based on variations of the Lagrangian $\mathcal{L}$. Consider now the variation in $\mathcal{L}$ due to variations in the control vector $u(t)$ for fixed times t_I and t_F:

$$\delta\mathcal{L} = \left[(\frac{\partial\varphi}{\partial x} - \lambda^T)\delta x\right]_{t=t_F} + [\lambda^T \delta x]_{t=t_I} + \int_{t_I}^{t_F} \left[(\frac{\partial\mathcal{H}}{\partial x} + \dot{\lambda}^T)\delta x + \frac{\partial\mathcal{H}}{\partial u}\delta u\right] dt. \tag{6}$$

Since it would be tedious to determine the variations $\delta x(t)$ produced by a given $\delta u(t)$, the multiplier functions $\lambda(t)$ are chosen so that coefficients δx vanish in equation (6):

$$\frac{d\lambda}{dt} = -\frac{\partial\mathcal{H}}{\partial x} = -\frac{\partial\mathcal{L}}{\partial x} - \lambda^T \frac{\partial f}{\partial x}, \tag{7}$$

with boundary conditions

$$\lambda(t_F) = \left[\frac{\partial\varphi}{\partial x}\right]_{t=t_F}. \tag{8}$$

Then, equation (6) become

$$\delta\mathcal{L} = [\lambda^T \delta x]_{t=t_I} + \int_{t_I}^{t_F} \left[\frac{\partial\mathcal{H}}{\partial u}\delta u\right] dt. \tag{9}$$

For an extremum, $\delta\mathcal{L}$ must be zero for any arbitrary δu. This can only happen if:

$$\left[\frac{\partial\mathcal{H}}{\partial u}\right] = 0, \; t \in [t_I, t_F]. \tag{10}$$

Equations (7), (8), and (10) are known as the Euler-Lagrange equations in the calculus of variations.

Definition 2.3 (***Pontryagin's maximum principle***)**.** *A general expression of the necessary conditions of optimality for the unconstrained optimal control problem* (1) *is due to Pontryagin's maximum principle [3]:*

$$\mathcal{H}(t, x^*(t), u^*(t), \lambda^*(t)) = \max_{u(t) \in U} \mathcal{H}(t, x(t), u(t), \lambda(t)) \; \forall t \in I = [t_I, t_F]. \tag{11}$$

Early developments of the maximum principle were carried out by Pontryagin et al [3] and Hestens [15]. A good survey on maximum principles with several cases and extended to handle constraints in both control and state variables is due to Hartl, Sethi, and Vickson [16].

The necessary optimality conditions are derived from the maximum principle and can be also expressed based on the Euler-Lagrange equations:

Definition 2.4 (***Necessary optimality conditions***)**.** *The necessary optimality conditions for the unconstrained optimal control problem* (1) *which result from setting the first variation of the Lagrangian to zero, $\delta\mathcal{L} = 0$, are:*

$$\frac{d\lambda}{dt} = -\frac{\partial\mathcal{H}}{\partial x}, \quad \textit{adjoint equations;} \tag{12a}$$

$$\frac{\partial\mathcal{H},}{\partial u} = 0, \quad \textit{control equations;} \tag{12b}$$

$$\lambda(t_F) = \left[\frac{\partial\varphi}{\partial x}\right]_{t=t_F}, \quad \textit{transversality conditions;} \tag{12c}$$

$$\lambda(t_I) = 0, \quad \textit{transversality conditions.} \tag{12d}$$

They are referred to as the Euler-Lagrange equations.

The control equations (12b) are a simplified statement of the Pontryagin maximum principle. Notice that, in addition to the fulfillment of the Euler-Lagrange equations (12), necessary optimality conditions entails also the fulfillment of the set of differential equations (1b), final boundary conditions (1c) and initial conditions $x(t_I) = x_I$.

2.1.2. Constrained problems

Let us now briefly extend the necessary condition derived for the unconstrained problem to the case in which inequality and equality path constraints are considered [4].

Generalizing the unconstrained optimal control problem (1), let us assume that we also impose algebraic path constraints of the form

$$g[t, x(t), u(t)] = 0, \tag{13}$$

where the designator g is as in Problem (OCP) besides changes due to not including the vector of parameters p.

The treatment of path constraint (13) depends on the matrix of partial derivatives, $g_u = \frac{\partial g}{\partial u}$. Two possibilities exist: If the matrix g_u is full rank, then the set of differential equations 1b) and the set of algebraic equations (13) constitute a DAE dynamical system of index one, and the of equations (13) is termed as control variable equality constraint. For this case the Hamiltonian (3) is replaced by

$$\mathcal{H}(t, x(t), u(t), \lambda(t), \nu(t)) = L(t, x(t), u(t)) + \lambda^T(t) f(t, x(t), u(t)) + \nu^T(t) g[t, x(t), u(t)], \tag{14}$$

which will result in modification to both the adjoint equations (12a) and the control equations (12b). In this Hamiltonian (14) $\nu \in \mathbb{R}^{n_z}$ are adjoint variables associated to the equality constraints $g \in \mathbb{R}^{n_z}$.

The second possibility is that the matrix g_u is rank deficient. In this case we can differentiate the set of path constraints (13) with respect to t and reduce the index of the DAE system. The result is a new path constraint function, $\frac{dg}{dt} = \dot{g} = 0$. For this new path function, the matrix $\dot{g}_u$ may be full rank or rank deficient. If it is full rank, we operate as in the previous case, substituting matrix g_u for matrix $\dot{g}_u$. If it is rank deficient, the process must be repeated.

These inconveniences can also appear even in the absence of path constraints, when the so-called *singular arcs* appear. One expects the optimality condition $\frac{\partial\mathcal{H}^T}{\partial u} = \mathcal{H}_u^T = 0$ to define the control variable provided by the nonsingular matrix $\mathcal{H}_{uu}$. However, if $\mathcal{H}_{uu}$ is singular, the control u is not uniquely defined by the optimality condition. For this situation, referred to as singular arc, it holds an analysis of the problem involving techniques similar to those used for the case of equality constraints.

Let us now generalize the unconstrained optimal control problem (1) considering inequality path constraints of the form $\phi[t, x(t), u(t)] \geq 0$, where the designator ϕ is as in Problem (OCP) besides changes due to not including the vector of parameters p.

Unlike an equality constraint, which must be satisfied throughout the entire time domain $I = [t_I, t_F]$, inequality constraints may either be active ($\phi = 0$) or inactive ($\phi > 0$) at each instant in time. In essence, the time domain is partitioned into constrained and unconstrained subarcs. During the unconstrained arcs, the necessary conditions are given by the set of differential equations (1b), the set of adjoint equations (12a) and the set of control equations (12b), whereas the conditions with modified Hamiltonian (14) are applicable in the constrained arcs. Thus, the imposition of inequality constraints presents three major complications. First, te number of constrained subarcs is not known a priori. Second, the location of the junction points when the transition from constrained to unconstrained (and vice-versa) occurs is unknown. Finally, at the junction points, it is possible that both the control variables u and the adjoint variables λ are discontinuous. Additional jump conditions, which are essentially boundary conditions imposed at the junction points, must be satisfied.

For a more complete discussion on how a constraints are tackled in optimal control problems, the reader is referred to [17, chapter 3] and [4].

3. Numerical methods

Typically, optimal control problems are highly nonlinear and it is very difficult to find an analytical solution even for the simplest cases. The common practice is to use numerical methods to obtain solutions.

There are three main approaches to numerically solve continuos time optimal control problems such as problem (OCP):

1. **Dynamic Programming (DP) methods:** The optimality criteria in continuos time is based on the Hamilton-Jacobi-Belman partial differential equation [2].
2. **Indirect methods**: The fundamental characteristic is that they explicitly rely on the necessary conditions of optimality that can be derived from the Pontryagin's Maximum Principle [18]. Bryson and Ho [17] provide a thorough and comprehensive overview of necessary conditions for different types of unconstrained and constrained optimal control problems.
3. **Direct methods:** They can be applied without deriving the necessary condition of optimality. Direct methods are based on a finite dimensional parameterization of the infinite dimensional problem. The finite dimensional problem is typically solved using an optimization method, such as NLP techniques. NLP problems can be solved to local optimality relying on the so called Karush-Kuhn-Tucker (KKT) conditions, which give first-order conditions of optimality. These conditions were first derived by Karush in 1939 [19], and some years later, in 1951, independently by Kuhn and Tucker [20].

3.1. Dynamic programming methods

The basic idea in using DP is to subdivide the problem to be solved in a number of stages. Each stage is associated with one subproblem and the subproblems are linked together by a recurrence relation. The solution of the whole problem is thus obtained by solving the subproblems using recursive computations. For a more detailed insight in DP and optimal control, the reader is referred to [21].

DP has been extensively applied with success to discrete optimal control problems. Unfortunately, its application is severely restricted in the case of continuous states systems because of the "curse of dimensionality," a term coined by Bellman to describe the problem caused by the exponential increase in the size of the state space.

Therefore, for solving nonlinear, continuous optimal control problems with a large number of variables, e.g., the aircraft trajectory planning problem, DP is clearly not adequate. Other approaches, such as indirect or direct methods, must be used.

3.2. Indirect methods

Indirect methods rely on Pontryagin's Maximum Principle [18]. Typically, the optimal control problem is turned into a two point boundary value problem containing the same mathematical information as the original one by means of necessary conditions of optimality. Then, the boundary value problem is discretized by some numerical technique to get a solution. Thus, Indirect methods follow a "first optimize, then discretize" scheme. Numerical techniques for solving this two point boundary value problem can be classified as gradient methods [22], indirect shooting and indirect multiple shooting [23, 24], and indirect collocation [25].

The practical drawbacks of indirect methods are [4, Chap. 4.3], [26]:

- Proper formulations of the necessary conditions of optimality in a numerically suitable way must be derived. Since this formulation is rather complicated, significant knowledge and experience in optimal control is required by the user of an indirect method.
- In order to handle active constraints properly, their switching structure must be guessed.

- Suitable initial guesses of the state variables and, with special relevance, to the adjoint variables must be provided to start the iterative method. State variables have physical meaning, but adjoint variables do not, so that giving a proper initial guess might be hard and a non-proper one usually leads to non-optimal solutions. Even with a reasonable guess for the adjoint variables, the numerical solution of the adjoint equations can be ill conditioned.
- Changes in the problem formulation, e.g., by a modification of the model equations, imply formulating again the optimality conditions of the problem.
- Finally, model functions with low differentiability properties are difficult to tackle with indirect approaches.

Because of these practical difficulties, indirect methods are not suitable to solve highly constrained trajectory planning problems. In fact, rather than indirect approaches, direct methods have been extensively used for solving aerospace trajectory optimization problems in spite of the fact that they present less accuracy than indirect methods [27]. Two comprehensive surveys analyzing direct and indirect methods for trajectory optimization are [28, 29].

3.3. Direct methods

The so called direct methods do not use the first-order necessary conditions of the continuous optimal control problem. They convert the infinite dimensional problem into a problem with a finite set of variables, and then solve the finite dimensional problem using optimization methods. Direct methods thus follow a "first discretize, then optimize" approach. A typical strategy is to convert the infinite problem into a NLP problem which is solved using mathematical programming techniques [30, 31].

The most important direct numerical methods are direct shooting [32], direct multiple shooting [33] and direct collocation [34]. A good reference on the practical importance of direct methods is [4].

The direct single shooting method has been broadly used because it allows optimal control problems to be easily converted into an NLP problem with a small number of variables even for very large problems. In single shooting only initial guesses for the control NLP variables are required. In contrast, it is very sensitive to small perturbations on the initial condition.

The direct multiple shooting method reduces some of the problems that single shooting has. However, the multiple shooting approach increases the size of the problem because additional variables and constraints have to be included. When the problem includes inequality constraints, there is the additional disadvantage that the sequence of unconstrained and constrained arcs has to be specified in advance.

The direct collocation method do not suffer from most of the drawbacks mentioned above, and therefore they are the most suitable for aerospace trajectory optimization problems [4, 28, 29].

A taxonomy of optimal control methods for trajectory optimization is given in Figure 1. Notice that this taxonomy is not necessarily exhaustive.

3.3.1. Direct collocation methods

Collocation methods enforce the dynamic equations through quadrature rules or interpolation [35, 34]. A suitable interpolating function, or interpolant, is chosen such that it passes through the state values and maintains the state derivatives at the nodes spanning one interval, or subinterval, of time. The interpolant is then evaluated at points between nodes, called collocation points. At each collocation point, a constraint equating the interpolant derivative to the state derivative function is introduced to ensure that the equations of motion are approximately satisfied across the entire interval of time [36].

Collocation methods are characterized by the interpolating function and by the nodes and collocation points they use. One of the simplest methods of collocation is the Hermite-Simpson collocation method [35, 37]. In this method a third-order Hermite interpolating polynomial is used locally within the entire sequence of time subintervals, each solved at the endpoints of a subinterval and collocated at the midpoint. When arranged appropriately, the expression for the collocation constraint corresponds to the Simpson integration rule. A generalization of the method is obtained using the n-th order Hermite interpolating polynomial, and choosing the nodes and collocation points from a set of Legendre-Gauss-Lobatto points defined within the time subintervals. These choices give rise to the Hermite-Legendre-Gauss-Lobatto (HLGL) collocation method [36]. Other collocation methods are based, for instance, on Gauss or Radau collocation schemes [38, 39].

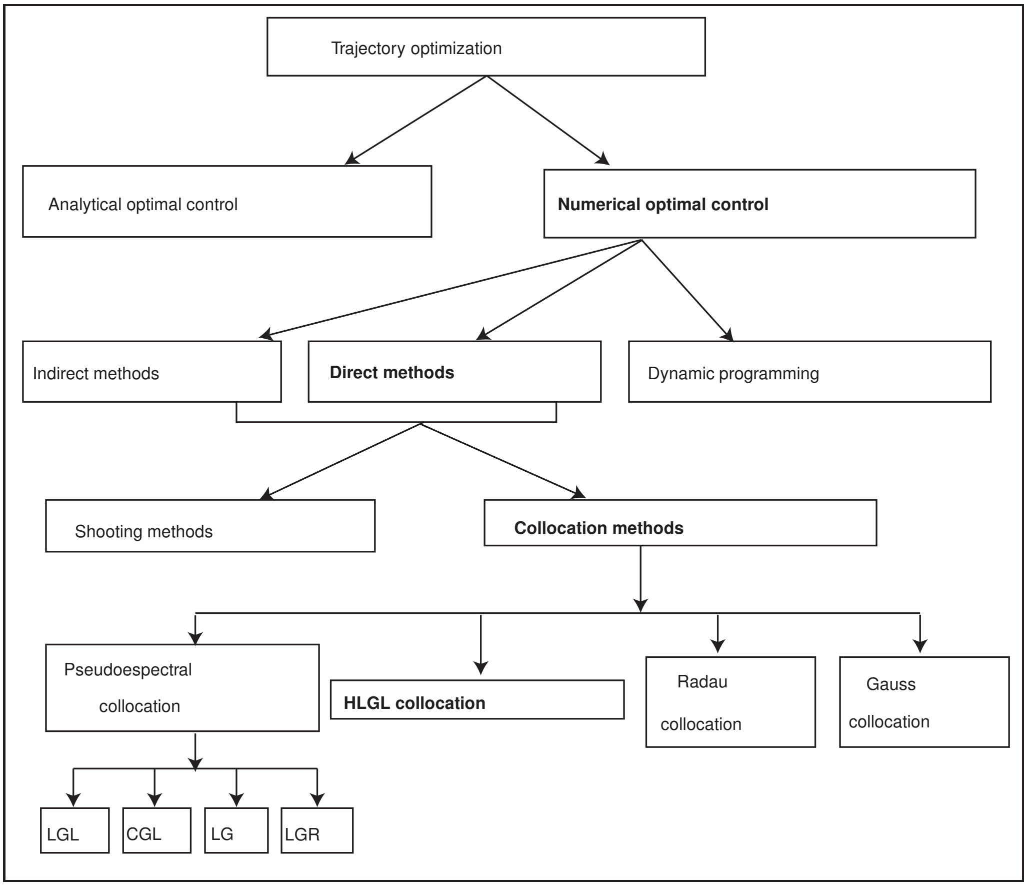


Figure 1. Taxonomy of trajectory optimization methods using optimal control.

There exist also discretizations for collocation based on pseudospectral methods, which generally use global orthogonal Lagrange polynomial as the interpolants while the nodes are selected as the roots of the derivative of the named polynomial, such as Legendre-Gauss-Lobatto (LGL) (Legendre pseudospectral collocation methods), Chebyshev-Gauss-Lobatto (CGL) (Chebyshev pseudospectral collocation methods), Legendre-Gauss (LG) (Gauss pseudospectral collocation methods), or Legendre-Gauss-Radau (LGR) (Radau pseudospectral collocation methods). Since these methods use global interpolants defined over the entire time interval, the Gauss-Lobatto nodes are clustered near the endpoints.

The reader is referred to [12, 13] and references therein for recent and comprehensive reviews of pseudospectal methods for optimal control.

4. Case study

Commercial aircraft during the cruise phase follow air routes that are composed of segments. Typically, due to ATM requirements aircraft should accomplish with a Required Time of Arrival (RTA) over a prescribed waypoint. The problem under analysis in this work is that of flying from one waypoint to another waypoint at a given flight level. A flat earth model is assumed, and therefore the segment can be considered a straight line. We want to find the optimal trajectory and the optimal control inputs given a RTA at the final waypoint.

4.1. Aircraft dynamics

4.1.1. Equations of motion

In order to plan optimal aircraft trajectories, it is common to consider a 3 degree of freedom dynamic model that describes the point variable-mass motion of the aircraft over a spherical flat-earth model. We consider a symmetric flight, that is, we assume there is no sideslip and all forces lie in the plane of symmetry of aircraft. Wind is not considered. The equations of motion of the aircraft are:

$$\frac{d}{dt}\begin{bmatrix} V \\ x_e \\ m \end{bmatrix} = \begin{bmatrix} \frac{T(t)-D(V(t),C_L(t))}{m(t)} \\ V(t) \\ -T(t)\cdot\eta(V(t)) \end{bmatrix}. \tag{15}$$

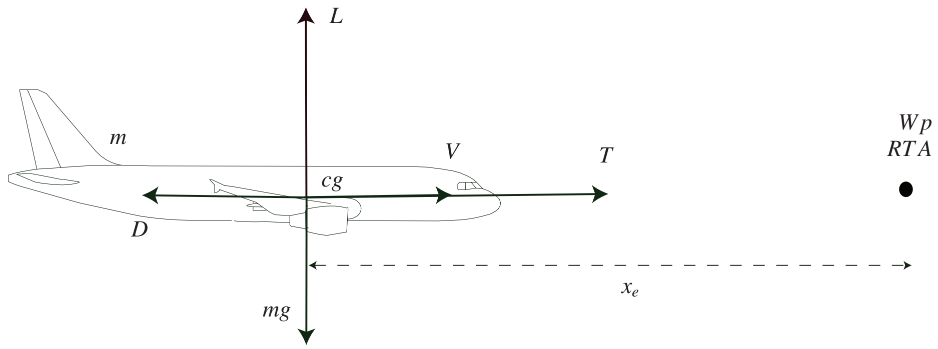


Figure 2. Aircraft state and forces

The states are: V, x_e, and m referring to the true airspeed, the longitudinal position, and the mass of the aircraft, respectively. η is the speed dependent fuel efficiency coefficient. Lift $L = C_L S\hat{q}$, which is equal to weight, and drag $D = C_D S\hat{q}$ are the components of the aerodynamic force, S is the reference wing surface area and $\hat{q} = \frac{1}{2}\rho V^2$ is the dynamic pressure. A parabolic drag polar $C_D = C_{D0} + KC_L^2$, and an International Standard Atmosphere (ISA) model are assumed. C_L is a known function of the angle of attack α and the Mach number. The engine thrust T is the input, that is, $u(t) = (T(t))$. For further details on aircraft dynamics, please refer to [40].

4.1.2. Flight envelope constraints

The flight envelope constraints are derived from the geometry of the aircraft, structural limitations, engine power, and aerodynamic characteristics. We use the BADA performance limitations model and parameters [41]. For this particular problem, we have:

$$\begin{array}{ll} M(t) \leq M_{M0}, & m_{min} \leq m(t) \leq m_{max}, \\ \dot{V}(t) \leq \bar{a}_l, & C_v V_s(t) \leq V(t) \leq V_{Mo}, \\ T_{min}(t) \leq T(t) \leq T_{max}(t), & 0 \leq C_L(t) \leq C_{L_{max}}. \end{array}$$

In the above, $M(t)$ is the Mach number and M_{M_0} is the maximum operating Mach number; C_v is the minimum speed coefficient, $V_s(t)$ is the stall speed and V_{M_0} is the maximum operating calibrated airspeed; $\bar{a}_l$ is the maximum longitudinal acceleration for civilian aircraft. T_{min} and T_{max} correspond, respectively, to the minimum and maximum available thrust. Note that several flight envelop constraints are nonconvex.

4.2. Aircraft data, boundary conditions, and objective function

For the analysis herein, we have selected an Airbus 320 BADA 3.9 model [41]. The aerodynamic parameters are those for cruise flap configuration.

The boundary conditions are shown in table 1. The objective functional is to minimize the total amount of fuel consumption.

Table 1. Boundary conditions.

States and control variables	Initial conditions	Final conditions
Time $t[s]$	0	4751
Longitude $x(t)[m]$	0	1000000
Velocity $V(t)[kts]$	420	Free
Mass $m(t)[kg]$	51200	Free
Thrust $T(t)[N]$	Free	Free

5. Numerical results

The previously described trajectory optimization problem is solved using 4 different methodologies, namely:

- Hermite-Simpson collocation method,
- 5th degree Gauss-Lobatto collocation method,
- Radau Psuedospectral collocation method using gpops, and
- Singular Arc solution.

The Hermite-Simpson collocation method has been implemented using a liner control interpolation scheme. The 5th degree Gauss-Lobatto collocation method has been implemented using a free control scheme. Both methods have been hand-tailored and implemented in AMPL modeling language. IPOPT was used as NLP solver. More information on these Gauss-Lobatto collocation methods can be found in [11].

GPOPS implements a Radau Psuedospectral collocation method [42]. It has been configure using SNOPT solver and the following setup information:

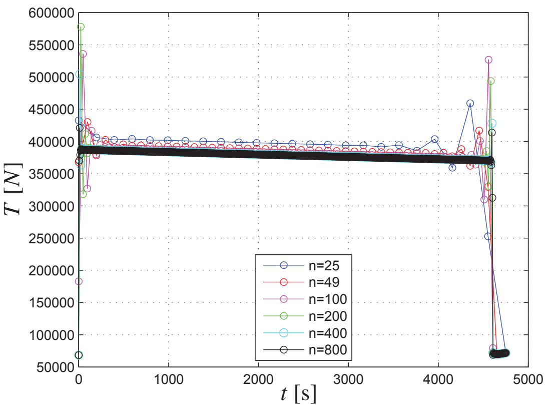
- setup.mesh.tolerance: 10^{-6},
- setup.mesh.iteration: 10,
- setup.autoscale: 'on',
- setup.derivatives: 'finite-difference',
- setup.checkDerivatives: 0,
- setup.maxIterations: 3000,
- setup.tolerances: $[10^{-6}, 2 \cdot 10^{-6}]$.

The singular arc analytic solution is that by Franco et al. in [5]. The solution is obtained using the ODE 45 function in Matlab.

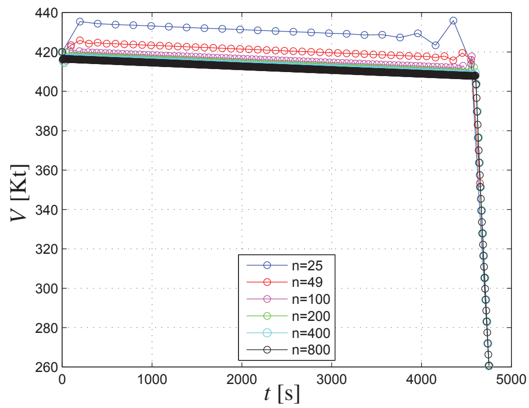
It is interesting to point out that it has been shown that the KKT NLP necessary conditions approach the optimal control necessary conditions of optimality as the number of variables grows. Indeed, at the solution of the NLP problem, the Lagrange multipliers can be interpreted as discrete approximations to the optimal control adjoint variables [4]. Therefore, we first solve the problem using the above mentioned direct methods for an increasing number of variables. The results are shown in Table 2, Table 3, and Table 4. Figure 3 depicts the different solutions for the true airspeed and the thrust.

For the Gauss-Lobatto methods, i.e., Hermite-Simpson and 5th degree, it can be observed that the objective function is very sensitive to the number of samples. This is specially noticeable for a low number of samples. The computational time is however very low. Notice that by increasing the number of samples, the solution converges to a value of minimum fuel consumption (objective function) and so do velocity and thrust. Moreover, the instabilities due to the boundary conditions are soften.

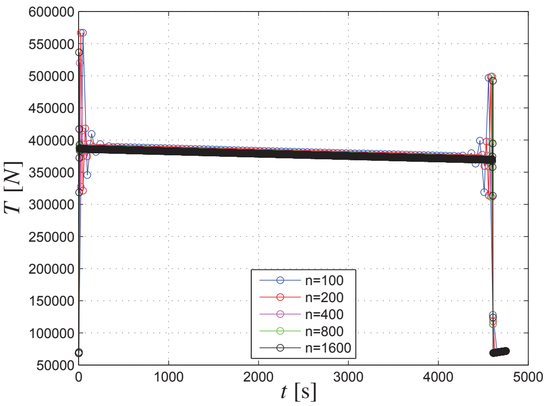
On the contrary, the gpops solution does not show much sensitivity to the number of samples. This is due to the fact that pseudospectral methods employ a non-homogeneous mesh of samples, locating more sample points in those regions with high dynamics, e.g., near the boundaries. Therefore, a fairly good solution can be obtained with much less number of variables. Nonetheless, the computation is intense more than ten times higher due to the matlab interface, which slows the process substantially, as it is shown in tables from 2 to 4.



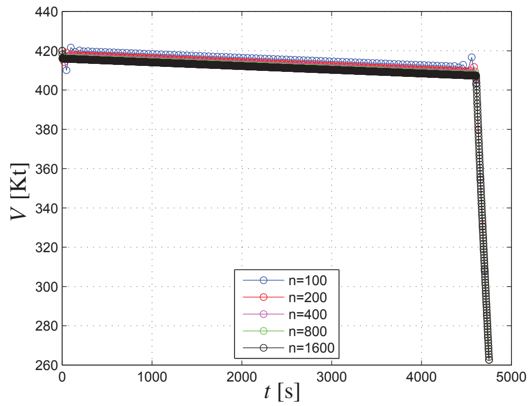
(a) 5th degree



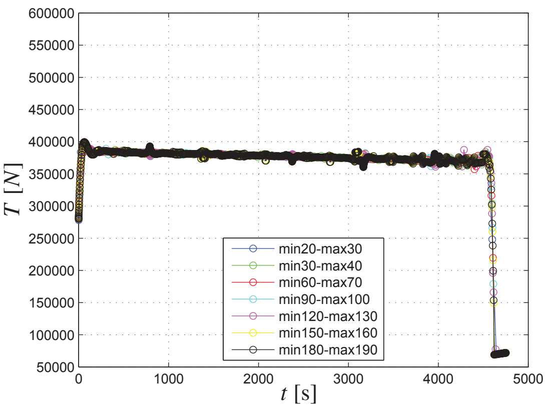
(b) 5th degree



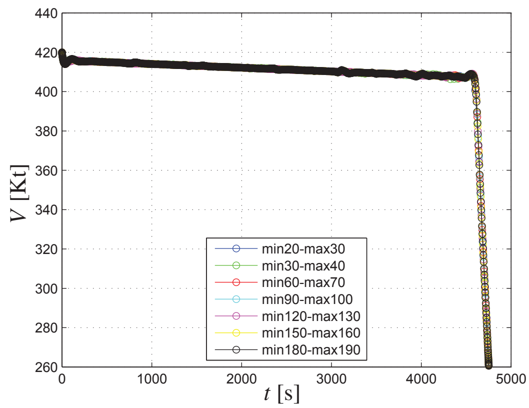
(c) Hermite-Simpson



(d) Hermite-Simpson



(e) Gpops



(f) Gpops

Figure 3. State variable $V(t)$ and control input $T(t)$ for different number of sample points

Singular arc solution provides 2507.7 kg of fuel consumption. This solution, even though is very smooth and thus nice from an operational perspective, results less efficient than any of the ones provided by the different direct methods. This is because the singular arc solution, which is indeed derived from Pontryagin's maximum principle, does not consider inequality constraints. Indeed, direct methods suggest that the optimal performance is to stall the aircraft when reaching the final waypoint at the RTA. This is obviously unrealistic, but for instance, at top of descent an aircraft might want to decelerate to intercept the waypoint at the optimal descent speed.

Table 2. Numerical results Hermite-Simpson.

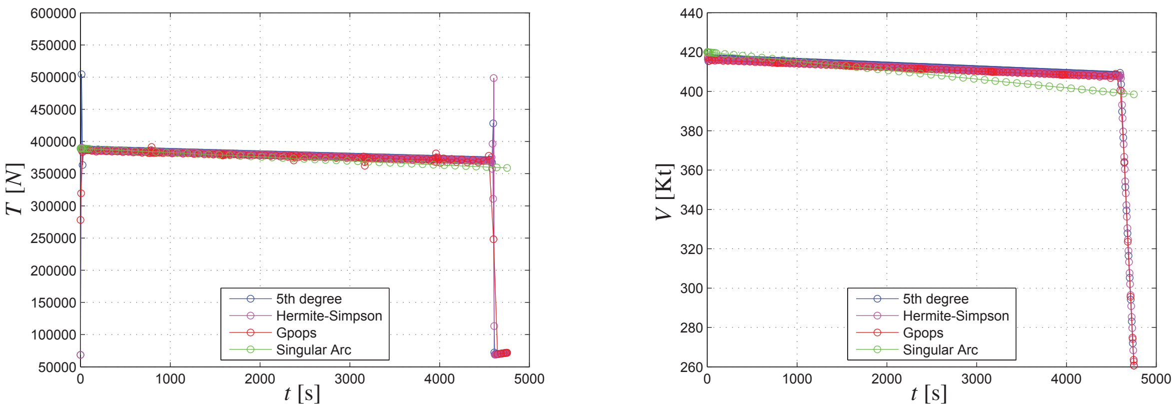
n	100	200	400	800	1600	3200
Fuel Consumption [kg]	2478.31	2474.34	2472.53	2471.67	2471.25	2471.04
CPU time in IPOPT [s]	0.044	0.098	0.195	0.402	0.888	2.246
CPU time in IPOPT NLP [s]	0.092	0.273	0.749	1.656	3.755	8.206
Total Time [s]	0.160	0.414	1.02501	2.2197	4.94	11.10
Iterations	17	20	20	21	23	25
Total number of variables	495	995	1995	3995	7995	15995
Total number of eq. constraints	396	796	1596	3196	6396	12796
Total number of ineq. constraints	297	597	1197	2397	4797	9597

Table 3. Numerical results 5th degree.

n	25	49	100	200	400	800
Fuel Consumption [kg]	2481.03	2463.75	2455.94	2452.53	2450.91	2450.13
CPU time in IPOPT [s]	0.083	0.188	0.438	167.35	2.076	4.012
CPU time in IPOPT NLP [s]	0.153	0.360	1.048	4.72	6.217	12.305
Total Time [s]	0.26	0.5859	1.56	172.41	8.58	16.90
Iterations	29	29	34	58	38	37
Total number of variables	433	865	1783	3583	7183	14383
Total number of eq. constraints	336	672	1386	2786	5586	11186
Total number of ineq. constraints	191	383	791	1591	3191	6396

Table 4. Numerical results gpops.

nodesPerInterval.min-nodesPerInterval.max	20 – 30	30 – 40	60 – 70	90 – 100	120 – 130	150 – 160	180 – 190
Fuel Consumption [kg]	2466.87	2466.86	2466.87	2466.87	2466.87	2466.87	2466.87
Total Time [s]	44.7	58.5	132.2	302.3	769.2	1249.8	2466.9
Iterations	369	1136	825	1026	1490	2297	2802
Number Mesh Refinement	1	1	1	1	1	1	1
Sample points	121	181	361	541	721	901	1081
Total number of variables	485	725	1445	2165	2885	3605	4325
Total number of linear constraints	2	2	2	2	2	2	2
Total number of NL constraints	360	540	1080	1620	2160	2700	3240



(a) Thrust vs time (b) Velocity vs time

Figure 4. State variable $V(t)$ and control input $T(t)$ for Singular arc, gpops, Hermite-Simpson and 5th degree simulations

Thus, the singular arc solution cannot be claimed as the solution to the indirect method. More efforts are needed to find the indirect solution, which would provide a baseline to compare the quality of the different methods.

Figure 4 shows the comparative of the four methods. Aircraft weight force decreases with time, aircraft lift force is considered equal to aircraft weight force and aircraft drag force is evaluated making use of the parabolic drag polar, therefore velocity also decrease slightly with time and due to the fact that the cost function is the aircraft fuel consumption then the aircraft thrust force decrease with time. It is important to observe that at the end of the solution the thrust is going to the minimum value, thus the velocity is close to the stall value to save fuel.

6. Conclusion

This study the performances of four trajectory optimization methods: Hermite-Simpson, 5th degree Gauss-Lobatto, Radau Pseudospectral collocation methods using goops and singular arc solution, to solve a constant altitude and course flight with fixed arrival time, ISA atmosphere and without wind.

5th degree Gauss-Lobatto collocation method provides the lest fuel consumption solution but Hermite-Simpson collocation method is the fastest with a solution no so far from the 5th degree.

Gpops is a user friendly matlab tool, with few programming tasks a good solution is accomplished. However, the collocation methods Hermite-Simpson and 5th degree have been programmed using AMPL modeling language which needs more programming effort but once the codification is done they produce fast and good solutions.

Singular arc solution is flyable procedure due to the thrust is almost constant and the velocity is constantly decreasing. Nevertheless, this method does not produce the most optimal solution.

The future goals are to program a pseudospectral method , a dynamic programming method and the indirect method in AMPL modeling language and also to compare all these method in vertical or/and horizontal profile flight in order to evaluate the performance of them.

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